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D-boundedness and *D*-compactness in finite dimensional probabilistic normed spaces

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Abstract. In this paper, we prove that in a finite dimensional probabilistic normed space, every two probabilistic norms are equivalent and we study the notion of *D*-compactness and *D*-boundedness in probabilistic normed spaces.

Keywords. Probabilistic normed space; finite dimensional normed space; *D*-bounded; *D*-compact.

1. Introduction and preliminaries

K Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [8]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. Probabilistic normed spaces were introduced by Šerstnev [9] in 1962 by means of a definition that was closely modelled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. In the sequel, we shall adopt the usual terminology, notation and conventions of the theory of probabilistic normed spaces, as in [1,2,4–8,10].

In the sequel, the space of probability distribution functions (briefly, d.f.) is $\Delta^+ = \{F\colon \mathbf{R} \to [0,1]\colon F$ is left-continuous, nondecreasing, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set

$$D^+ = \{F \in \Delta^+\colon l^- F(+\infty) = 1\}.$$

Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(x) \leq G(x)$, for all x in $\mathbf{R}$. The maximal element for Δ^+ in this order is the

d.f. given by

$$\varepsilon_0 = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A triangle function is a binary operation on Δ^+, namely a function τ: $\Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, that is

$$\begin{aligned} &\tau(\tau(F,G),H) = \tau(F,\tau(G,H)), \\ &\tau(F,G) = \tau(G,F), \\ &F \le G \Longrightarrow \tau(F,H) \le \tau(G,H), \\ &\tau(F,\varepsilon_0) = F \end{aligned}$$

for all $F,G,H \in \Delta^+$. Continuity of triangle functions means continuity with respect to the topology of weak convergence in Δ^+.

Typical continuous triangle functions are $\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t))$ and $\tau_{T^*}(F,G) = \inf_{s+t=x} T^*(F(s),G(t))$. Here T is a continuous t-norm, i.e., a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$.

The definition below is more general, it has been proposed in [1].

DEFINITION 1.

A *probabilistic normed space* (briefly, a PN space) is a quadruple (V,ν,τ,τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \le \tau^*$ and ν is a mapping (the *probabilistic norm*) from V into Δ^+, such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in V);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \ge \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \le \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A PN space is called a *Šerstnev space* if it satisfies (N1), (N3) and the following condition:

For every $\alpha \ne 0 \in \mathbf{R}$ and $x > 0$ one has

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right),$$

which clearly implies (N2) and also (N4) in the strengthened form

$$\forall \lambda \in [0,1], \quad \nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}).$$

A PN space in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for a suitable continuous t-norm T and its t-conorm T^* is called a *Menger PN space*.

Lemma 2. [2]. *If* $|\alpha| \le |\beta|$, *then* $\nu_{\beta p} \le \nu_{\alpha p}$ *for every* p *in* V.

Lemma 3. [2]. *If τ^* is Archimedean, then for every p in V such that $\nu_p \neq \varepsilon_\infty$ and every $h > 0$, there is a $\delta > 0$ such that*

$$|\alpha| < \delta \Longrightarrow \nu_{\alpha p}(h) > 1 - h.$$

In this case PN space is a topological vector space (shortly, TVS). We call every PN space with the above properties as *strong TVS*.

DEFINITION 4. [8].

Let (V, ν, τ, τ^*) be a PN space. For each p in V and $\lambda > 0$, the strong λ*-neighborhood* of p is the set

$$N_p(\lambda) = \{q \in V\colon \nu_{p-q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for V is the union $\cup_{p\in V}\mathscr{N}_p$ where $\mathscr{N}_p = \{N_p(\lambda)\colon \lambda > 0\}$.

The strong neighborhood system for V determines a Hausdorff topology for V.

DEFINITION 5. [8].

Let (V, ν, τ, τ^*) be a PN space, a sequence $\{p_n\}_n$ in V is said to be strongly convergent to p in V if for each $\lambda > 0$, there exists a positive integer N such that $p_n \in N_p(\lambda)$, for $n \geq N$. Also the sequence $\{p_n\}_n$ in V is called strongly Cauchy sequence if for every $\lambda > 0$ there is a positive integer N such that $\nu_{p_n - p_m}(\lambda) > 1 - \lambda$, whenever $m, n > N$. A PN space (V, ν, τ, τ^*) is said to be strongly complete in the strong topology if and only if every strongly Cauchy sequence in V is strongly convergent to a point in V.

DEFINITION 6. [5].

Let (V, ν, τ, τ^*) be a PN space and A be the nonempty subset of V. The probabilistic radius of A is the function R_A defined on $\mathbf{R}^+$ by

$$R_A(x) = \begin{cases} l^- \inf\{\nu_p(x)\colon p \in A\}, & \text{if } x \in [0, +\infty), \\ 1, & \text{if } x = +\infty. \end{cases}$$

DEFINITION 7. [5].

A nonempty set A in a PN space (V, ν, τ, τ^*) is said to be:

(a) certainly bounded, if $R_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$;
(b) perhaps bounded, if one has $R_A(x) < 1$, for every $x \in (0, +\infty)$ and $l^- R_A(+\infty) = 1$;
(c) perhaps unbounded, if $R_A(x_0) > 0$ for some $x_0 \in (0, +\infty)$ and $l^- R_A(+\infty) \in (0, 1)$;
(d) certainly unbounded, if $l^- R_A(+\infty) = 0$, i.e., if $R_A = \varepsilon_\infty$.

Moreover, A is said to be distributionally bounded, or simply D-bounded if either (a) or (b) holds, i.e., $R_A \in D^+$. If $R_A \in \Delta^+ \backslash D^+$, A is called D-unbounded.

Theorem 8 [5]. *A subset A in the PN space (V, ν, τ, τ^*) is D-bounded if and only if there exists a d.f. $G \in D^+$ such that $\nu_p \geq G$ for every $p \in A$.*

If $A \subset \mathbf{R}$ is D-bounded then in general A is not classically bounded.

Example 9. We consider the PN space $(\mathbf{R}, \nu, \tau, \mathbf{M})$, where τ is a triangle function such that $\tau(\varepsilon_c, \varepsilon_d) \le \varepsilon_{c+d}$, $\mathbf{M}$ is the maximal triangle function and the probabilistic norm ν: $\mathbf{R} \to \mathbf{\Delta}^+$ is defined by $\nu_p = \varepsilon_{\frac{|p|}{a+|p|}}$ for every p in $\mathbf{R}$ and for a fixed $a > 0$, with $\nu_p(+\infty) = 1$ (see Theorem 5 of [3]). With this norm, $\mathbf{R}$ is D-bounded because $\nu_p \ge \varepsilon_1$.

DEFINITION 10.

We say that the probabilistic norm ν: $\mathbf{R} \to \mathbf{\Delta}^+$ has the Lafuerza Guillén property (briefly, the LG-property) if, for every $x > 0$, $\lim_{p\to\infty} \nu_p(x) = 0$, or, equivalently, $\lim_{p\to\infty} \nu_p = \varepsilon_\infty$.

Example 11. The probabilistic norm in the last example does not have the LG-property.

Example 12. The quadruple $(\mathbf{R}, \nu, \tau_\pi, \tau_\pi^*)$, where ν: $\mathbf{R} \to \mathbf{\Delta}^+$ is defined by

$$\nu_p(x) = \begin{cases} 0, & \text{if } x = 0, \\ \exp(-|p|^{1/2}), & \text{if } 0 < x < +\infty, \\ 1, & \text{if } x = +\infty, \end{cases}$$

and $\nu_0 = \varepsilon_0$ is a PN space (see [1]) but is not Šerstnev space and the probabilistic norm has the LG-property.

Lemma 13. *In a PN space* $(\mathbf{R}, \nu, \tau, \tau^*)$ *in which the probabilistic norm has the LG-property, if* $A \subset \mathbf{R}$ *is D-bounded then it is classically bounded.*

Proof. If $A \subset \mathbf{R}$ is D-bounded, there exists a d.f. $G \in D^+$ such that $\nu_p \ge G$, for every $p \in A$ but if A is not classically bounded, then for every $k > 0$ there exists a $p \in A$ such that $|p| > k$. Hence $\lim_{p\to\infty} \nu_p(x) = 0$. Therefore for every $x \in (0, +\infty)$ we have $G(x) = 0$, which is a contradiction. □

The converse of the above lemma is, in general, not true. See Example 12. Here the only D-bounded set is the singleton $\{\theta\}$.

Theorem 14. *If the PN space* $(\mathbf{R}, \nu, \tau, \tau^*)$ *is a TVS then it is complete.*

Proof. Let $\{p_m\}$ be strongly Cauchy sequence, for every $m, n \in N$ $m > n$ we have

$$\lim_{m,n\to\infty} \nu_{p_m - p_n} = \varepsilon_0.$$

By TVS property we have

$$\nu_{\lim_{m,n}(p_m - p_n)} = \varepsilon_0 = \nu_0.$$

Hence $\{p_m\}$ is a classical Cauchy sequence in $\mathbf{R}$; therefore, it is convergent to $p \in \mathbf{R}$, i.e., $p_m - p \to 0$, since the PN space is TVS we have $\lim_m \nu_{p_m - p} = \nu_0 = \varepsilon_0$. □

2. Finite dimensional PN space

In this section, we are interested in some properties of a finite dimensional PN space, in particular we introduce the definition of equivalent norms in a PN space.

Theorem 15. *Let $\{p_1,\dots,p_n\}$ be a linearly independent set of vectors in a PN space (V,ν,τ,τ^*) such that τ^* is Archimedean and $\nu_p \neq \varepsilon_\infty$, for every $p\in V$. Then there is a number $c\neq 0$ and there exists a probabilistic norm ν': $\mathbf{R}\to\mathbf{\Delta}^+$ on the real PN space $(\mathbf{R},\nu',\tau',\tau'^*)$ where τ'^* is Archimedean and $\nu'_p\neq\varepsilon_\infty$, such that for every choice of n real scalars $\alpha_1,\dots,\alpha_n$ we have*

$$\nu_{\alpha_1 p_1+\cdots+\alpha_n p_n} \leq \nu'_{c(|\alpha_1|+\cdots+|\alpha_n|)}. \tag{1}$$

Proof. We write $s=|\alpha_1|+\cdots+|\alpha_n|$. If $s=0$, all α_j are zero, so (1) holds. Let $s>0$. We define $\mu_p=\nu_{sp}$ and $\mu'_r=\nu'_{sr}$. Then (1) is equivalent to the following inequality,

$$\mu_{\beta_1 p_1+\cdots+\beta_n p_n} \leq \mu'_c, \quad \beta_j=\alpha_j/s, \left(\sum_{j=1}^n|\beta_j|=1\right). \tag{2}$$

Hence it suffices to prove the existence of $c\neq 0$ and μ' such that (2) holds. Suppose otherwise, then there exists a sequence $\{q_m\}$ of vectors

$$q_m=\beta_1^{(m)}p_1+\cdots+\beta_n^{(m)}p_n, \left(\sum_{j=1}^n|\beta_j^{(m)}|=1\right),$$

such that $\mu_{q_m}\to\varepsilon_0$ as $m\to\infty$. Since $\sum_{j=1}^n|\beta_j^{(m)}|=1$, we have $|\beta_j^{(m)}|\leq 1$. Hence, the sequence $\{\beta_1^{(m)}\}$ has a convergent subsequence. Let β_1 denote the limit of such a subsequence, and let $\{q_{1,m}\}$ denote the corresponding subsequence of $\{q_m\}$. By the same argument, $\{q_{1,m}\}$ has a subsequence $\{q_{2,m}\}$ for which the corresponding sequence of real scalars $\{\beta_2^{(m)}\}$ converges say to β_2. Continuing this process, we obtain a subsequence $\{q_{n,m}\}$ of $\{q_m\}$ such that

$$q_{n,m}=\sum_{j=1}^n\gamma_j^{(m)}p_j, \left(\sum_{j=1}^n|\gamma_j^{(m)}|=1\right)$$

and $\gamma_j^{(m)}\to\beta_j$ as $m\to\infty$. Hence

$$\lim_{m\to\infty}q_{n,m}=q:=\sum_{j=1}^n\beta_j p_j,$$

where $\sum_{j=1}^n|\beta_j|=1$, since

$$\begin{aligned}\mu_{q_{n,m}-q} &= \mu_{\sum_{j=1}^n(\gamma_j^{(m)}-\beta_j)p_j}\\ &\geq \tau^n(\mu_{(\gamma_1^{(m)}-\beta_1)p_1},\dots,\mu_{(\gamma_n^{(m)}-\beta_n)p_n})\to\varepsilon_0\end{aligned}$$

as $m\to\infty$. Since $\{p_1,\dots,p_n\}$ is linearly independent and not all β_j's are zero, we have $q\neq\theta$. Since $\mu_{q_m}\to\varepsilon_0$, we have $\mu_{q_{n,m}}\to\varepsilon_0$. But,

$$\mu_q=\mu_{(q-q_{n,m})+q_{n,m}}\geq\tau(\mu_{q-q_{n,m}},\mu_{q_{n,m}})\to\varepsilon_0,$$

and hence $q=\theta$. This contradicts $q\neq\theta$. □

The following example shows that in the above theorem we need the field $\mathbf{R}$ to be a strong TVS.

Example 16. Consider the PN space $(\mathbf{R}, \nu, \tau, \tau^*)$ where τ^* is Archimedean and $\nu_p \neq \varepsilon_\infty$. By the above theorem there exists a $c \neq 0$ and a probabilistic norm $\nu'\colon \mathbf{R} \to \mathbf{\Delta}^+$ such that $\nu_p \leq \nu'_{cp}$. If in the PN space $(\mathbf{R}, \nu', \tau', \tau'^*)$ $\lim_m \nu'_{p_m} < \varepsilon_0$ whenever $p_m \to 0$ in $\mathbf{R}$, then for the sequence $\{2^{-n}\}$ we have $\nu_{2^{-n}} \leq \nu'_{c2^{-n}}$ and consequently $\varepsilon_0 < \varepsilon_0$, which is a contradiction.

From now on all the fields are strong TVS.

Theorem 17. *Every finite dimensional subspace W of a PN space (V, ν, τ, τ^*) where τ^* is Archimedean and $\nu_p \neq \varepsilon_\infty$ for every $p \in V$, is complete. In particular, every finite dimensional PN space is complete.*

Proof. Let $\{q_m\}$ be a strong Cauchy sequence in W. Let $\dim W = n$ and $\{w_1, \ldots, w_n\}$ be a linearly independent subset of W. Then each q_m has a unique representation of the form

$$q_m = \alpha_1^{(m)} w_1 + \cdots + \alpha_n^{(m)} w_n.$$

Since $\{q_m\}$ is a strong Cauchy sequence, for every $h > 0$ there is a positive integer N such that

$$\nu_{q_m - q_k}(h) > 1 - h$$

whenever $m, k \geq N$. By the above theorem and Lemma 2, we have, for every $j = 1, 2, \ldots, n$,

$$\begin{aligned} 1 - h &< \nu_{q_m - q_k}(h) \\ &= \nu_{\sum_{j=1}^n (\alpha_j^{(m)} - \alpha_j^{(k)}) w_j}(h) \\ &\leq \nu'_{c \sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j^{(k)}|}(h) \\ &\leq \nu'_{c|\alpha_j^{(m)} - \alpha_j^{(k)}|}(h), \end{aligned}$$

where $c \neq 0$, $\nu'\colon \mathbf{R} \to \mathbf{\Delta}^+$ and $m, k \geq N$. This shows that each of the n sequences $\{\alpha_j^{(m)}\}_m$ where $j = 1, 2, \ldots, n$ is a strong Cauchy in $(\mathbf{R}, \nu', \tau', \tau'^*)$. Hence it converges, say to α_j. Now let us define $q = \alpha_1 w_1 + \cdots + \alpha_n w_n$. Clearly, $q \in W$. Furthermore,

$$\begin{aligned} \nu_{q_m - q} &= \nu_{\sum_{j=1}^n (\alpha_j^{(m)} - \alpha_j) w_j} \\ &\geq \tau^n (\nu_{(\alpha_1^{(m)} - \alpha_1) w_1}, \ldots, \nu_{(\alpha_n^{(m)} - \alpha_n) w_n}), \end{aligned}$$

tends to ε_0 whenever $m \to \infty$. That is q_m strongly converges to q. Hence W is complete. □

DEFINITION 18.

A probabilistic norm $\nu\colon V \to \Delta^+$ is said to be equivalent to a probabilistic norm $\mu\colon V \to \Delta^+$, if $p_m \xrightarrow{\nu} p$ is equivalent to $p_m \xrightarrow{\mu} p$.

In the following example two equivalent norms in probabilistic normed space are given.

Example 19. We consider two PN space (V,ν,τ,τ^*) with probabilistic norm $\nu_p=\varepsilon_{\|p\|}$ where $\tau(\varepsilon_c,\varepsilon_d)=\varepsilon_{c+d},(c>0,d>0)$ and $\tau\le\tau^*$ (see Example 1 of [5]) and $(V,\mu,\tau',\mathbf{M})$ with probabilistic norm $\mu_p=\varepsilon_{\frac{\|p\|}{a+\|p\|}}, a>0$ (see Theorem 5 of [3]). It is easy to see that the two probabilistic norms are equivalent and so identity map $J\colon V\to V$ is continuous.

Theorem 20. *If V is a finite dimensional vector spaces, then every two probabilistic norms ν of (V,ν,τ,τ^*) and μ of (V,μ,τ',τ'^*) are equivalent, whenever τ^* and τ'^* are Archimedean, $\nu_p\neq\varepsilon_\infty$, and $\mu_p\neq\varepsilon_\infty$, for every p in V.*

Proof. Let $\{v_1,\ldots,v_n\}$ be a linearly independent subset of V. Let $p_m\xrightarrow{\nu}p$. We know that both p_m and p have a unique representation as

$$p_m=\alpha_1^{(m)}v_1+\cdots+\alpha_n^{(m)}v_n,$$

and $p=\alpha_1v_1+\cdots+\alpha_nv_n$. By Theorem 15 and Lemma 2, we have

$$\nu_{p_m-p}=\nu_{\sum_{j=1}^n(\alpha_j^{(m)}-\alpha_j)v_j}\le\nu'_{c\sum_{j=1}^n|\alpha_j^{(m)}-\alpha_j|}\le\nu'_{c|\alpha_j^{(m)}-\alpha_j|},$$

where $c\neq0$ and $\nu'\colon\mathbf{R}\to\mathbf{\Delta}^+$. Therefore $\lim_m\nu'_{c(|\alpha_j^{(m)}-\alpha_j|)}=\nu'_{c(\lim_m|\alpha_j^{(m)}-\alpha_j|)}=\varepsilon_0$, that is $\alpha_j^{(m)}\to\alpha_j$ in $\mathbf{R}$. But

$$\mu_{p_m-p}=\mu_{\sum_{j=1}^n(\alpha_j^{(m)}-\alpha_j)v_j}\ge\tau'^n(\mu_{(\alpha_1^{(m)}-\alpha_1)v_1},\ldots,\mu_{(\alpha_n^{(m)}-\alpha_n)v_n}),$$

so by continuity of τ' we have $p_m\xrightarrow{\mu}p$. By the same argument $p_m\xrightarrow{\mu}p$ implies $p_m\xrightarrow{\nu}p$. □

In the next example we show that there are two PN spaces which are not equivalent even in a finite dimensional probabilistic normed space. Indeed, since τ_M is not Archimedean, the first PN space in the next example is not strong TVS.

Example 21. We consider PN space $(\mathbf{R},\nu,\tau_{\mathbf{W}},\tau_{\mathbf{M}})$ where $\nu_0=\varepsilon_0$ and $\nu_p=\frac{1}{|p|+2}\varepsilon_0+\frac{|p|+1}{|p|+2}\varepsilon_\infty$ for $p\neq0$, we know τ_M is not Archimedean, (see [2]) and PN space $(\mathbf{R},\nu,\tau,\tau^*)$ with probabilistic norm $\nu_p=\varepsilon_{|p|}$ where $\tau(\varepsilon_c,\varepsilon_d)=\varepsilon_{c+d},(c>0,d>0)$ and $\tau\le\tau^*$ (see Example 1 of [5]). Now, the sequence $\{1/n\}$ in the first PN space is not convergent but in the second it is convergent. Therefore the above PN spaces are not equivalent.

3. *D*-bounded and *D*-compact sets in PN spaces

Theorem 22. *Let (V,ν,τ,τ^*) be a PN space in which $\nu(V)\subseteq D^+$ and D^+ is invariant under τ, i.e., $\tau(D^+\times D^+)\subseteq D^+$. If $p_m\to p$ in V and $A=\{p_m\colon m\in\mathbf{N}\}$, then A is a D-bounded subset of V.*

Proof. Let $p_m\to p$. Then there exists a positive integer N such that for every $m\ge N$ we have $\nu_{p_m-p}\ge G$, for each $G\in D^+$. Therefore

$$\nu_{p_m}\ge\tau(\nu_{p_m-p},\nu_p)\ge\tau(G,\nu_p).$$

If we put $H=\min\{\nu_{p_1},\ldots,\nu_{p_{N-1}},\tau(G,\nu_p)\}$, then $H\in D^+$ and $\nu_{p_m}\ge H$, for every $m\in\mathbf{N}$. Hence A is a D-bounded set. □

Note that, in the Example 12 in which $\nu(V) \subseteq \Delta^+ \backslash D^+$, the sequence $\left\{\frac{1}{m}\right\}$ is convergent but $A = \left\{\frac{1}{m} \colon m \in \mathbf{N}\right\}$ is not a D-bounded set.

DEFINITION 23.

The PN space (V, ν, τ, τ^*) is said to be *distributionally compact* (simply D-compact) if every sequence $\{p_m\}_m$ in V has a convergent subsequence $\{p_{m_k}\}$. A subset A of a PN space (V, ν, τ, τ^*) is said to be D-compact if every sequence $\{p_m\}$ in A has a subsequence $\{p_{m_k}\}$ convergent to a vector $p \in A$.

Lemma 24. *A D-compact subset of a PN space (V, ν, τ, τ^*) in which $\nu(V) \subseteq D^+$ and D^+ is invariant under τ, is D-bounded and closed.*

Proof. Suppose that $A \subseteq V$ is D-compact. If A is D-unbounded, it contains a D-unbounded sequence $\{p_m\}$ such that $\nu_{p_m} < \varepsilon_m$. This sequence could not have a convergent subsequence, since a convergent sequence must be D-bounded by Theorem 22. The closedness of A is trivial. □

As in the classical case, a D-bounded and closed subset of a (finite dimensional) PN space is not D-compact in general, as one can see from the next examples.

Example 25. We consider quadruple $(\mathbf{Q}, \nu, \tau_\pi, \tau_{\mathbf{t_2}})$, where $\pi(a,b) = a.b$, $t_2(a,b) = \frac{1}{1+[((1/a)-1)^2+((1/b)-1)^2]^{1/2}}$, for every $a, b \in (0,1)$ and probabilistic norm $\nu_p(t) = \frac{t}{t+|p|^{1/2}}$. It is straightforward to check that $(\mathbf{Q}, \nu, \tau_\pi, \tau_{\mathbf{t_2}})$ is a PN space. In this space, convergence of a sequence is equivalent to its convergence in $\mathbf{R}$. We consider the subset $A = [a,b] \cap \mathbf{Q}$, where $a, b \in \mathbf{R} \backslash \mathbf{Q}$. Since $R_A(t) = \frac{t}{t+(\max\{|a|,|b|\})^{1/2}}$, then A is a D-bounded set and since A is closed in $\mathbf{Q}$ classically, and so is closed in $(\mathbf{Q}, \nu, \tau_\pi, \tau_{\mathbf{t_2}})$. We know A is not classically compact in $\mathbf{Q}$, i.e., there exists a sequence in $\mathbf{Q}$ with no convergent subsequence in a classical sense and so in $(\mathbf{Q}, \nu, \tau_\pi, \tau_{\mathbf{t_2}})$. Hence A is not D-compact.

Example 26. We consider the PN space introduced in Example 9. With this probabilistic norm, $\mathbf{R}$ is D-bounded and closed. But $\mathbf{R}$ is not D-compact, because the sequence $\{2^m\}$ in $\mathbf{R}$ does not have any convergent subsequence in this space.

Example 27. The quadruple $(\mathbf{R}, \nu, \tau, \mathbf{M})$, where $\mathbf{M}$ is the maximal triangle function, and the probabilistic norm is a map $\nu \colon \mathbf{R} \to \mathbf{\Delta}^+$ such that $\nu_0 = \varepsilon_0$, $\nu_p = \varepsilon_{\frac{a+|p|}{a}}$ if $p \neq 0$, $(a > 0)$, and $\tau(\varepsilon_c, \varepsilon_d) \leq \varepsilon_{c+d}, (c > 0, d > 0)$, is a PN space (see Theorem 4 of [3]). If A is a nonempty, classically bounded set in $\mathbf{R}$, then there exists $s > 0$ such that for every $p \in A$ we have $|p| \leq s$. Since $\nu_p \geq \varepsilon_{\frac{a+s}{a}}$, A is D-bounded. Also it is trivial that A is closed. Now we show that A is not D-compact. Assume, if possible, that A is D-compact and $\{p_m\}$ be an arbitrary sequence in A which has a subsequence $\{p_{m_k}\}$ convergent to some p in A, then we have

$$\lim_k \nu_{p_{m_k}-p} = \lim_k \varepsilon_{\frac{a+|p_{m_k}-p|}{a}} \neq \varepsilon_0.$$

This implies that p is not in A, a contradiction.

Theorem 28. *Consider a finite dimensional PN space (V, ν, τ, τ^*), where τ^* is Archimedean, $\nu_p \neq \varepsilon_\infty$, and $\nu(V) \subseteq D^+$ and D^+ is invariant under τ, for every $p \in V$ on the real field $(\mathbf{R}, \nu', \tau', \tau'^*)$, where ν' has the LG-property. Every subset A of V is D-compact if and only if A is D-bounded and closed.*

Proof. By Lemma 24, D-compact subsets A of V are D-bounded and closed, so we need to prove the converse. Let $\dim V = n$ and $\{w_1, \ldots, w_n\}$ be a linearly independent subset of V. We consider any sequence $\{q_m\}$ in A. Each q_m has a representation $q_m = \alpha_1^{(m)} w_1 + \cdots + \alpha_n^{(m)} w_n$. Since A is D-bounded so is $\{q_m\}$, and so there exists a d.f. $G \in D^+$ such that

$$\begin{aligned} G &\le \nu_{q_m} \\ &\le \nu_{\alpha_1^{(m)} w_1 + \cdots + \alpha_n^{(m)} w_n}. \end{aligned}$$

By Theorem 15 and Lemma 2, we have

$$\begin{aligned} &\le \nu'_{c(|\alpha_1^{(m)}| + \cdots + |\alpha_n^{(m)}|)} \\ &\le \nu'_{c|\alpha_j^{(m)}|}. \end{aligned}$$

Hence for each fixed j, the sequence $\{\alpha_j^{(m)}\}$ is D-bounded and since ν' has the LG-property then by Lemma 13, it is also classically bounded. Therefore for every $1 \le j \le n$ the sequence $\{\alpha_j^{(m)}\}$ has a convergent subsequence converging to some α_j, $j = 1, \ldots, n$. As in the proof of Theorem 15, we can construct a subsequence $\{r_m\}$ of $\{q_m\}$ which converges to $r := \sum_{j=1}^{n} \alpha_j w_j$. Since A is closed, $r \in A$. This shows that each sequence $\{q_m\}$ in A has a convergent subsequence in A. Hence A is D-compact. □

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References

[1] Alsina C, Schweizer B and Sklar A, On the definition of a probabilistic normed space, *Aequationes Math.* **46** (1993) 91–98

[2] Alsina C, Schweizer B and Sklar A, Continuity properties of probabilistic norms, *J. Math. Anal. Appl.* **208** (1997) 446–452

[3] Lafuerza Guillén B, *D*-bounded sets in probabilistic normed spaces and their products, *Rend. Mat., Serie VII* **21** (2001) 17–28

[4] Lafuerza Guillén B, Rodríguez Lallena J A and Sempi C, Some classes of probabilistic normed spaces, *Rend. Mat., Serie VII* **17** (1997) 237–252

[5] Lafuerza Guillén B, Rodríguez Lallena J A and Sempi C, A study of boundedness in probabilistic normed spaces, *J. Math. Anal. Appl.* **232** (1999) 183–196

[6] Lafuerza Guillén B, Rodríguez Lallena J A and Sempi C, Normability of probabilistic normed spaces, to appear in *J. Math. Anal. Appl.*; http://front.math.ucdavis.edu/math.GN/0408305

[7] Khamsi M A and Kreinovich V Y, Fixed point theorems for dissipative mappings in complete probabilistic metric spaces, *Math. Jap.* **44** (1996) 513–520

[8] Schweizer B and Sklar A, Probabilistic Metric Spaces (1983) (New York: Elsevier, North Holland)

[9] Šerstnev A N, On the motion of a random normed space, *Dokl. Akad. Nauk SSSR* **149** (1963) 280–283, English translation in *Soviet Math. Dokl.* **4** (1963) 388–390

[10] Sibley D A, A metric for weak convergence of distribution functions, *Rocky Mountain J. Math.* **1** (1971) 427–430